A hybrid system provides a robust alternative to a linear regulator

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Introduction

"Multiple contact" animal gaits, in which multiple legs touch down nearly simultaneously, can be modeled by ordinary differential equations (ODEs) with periodic solutions and vector fields which are smooth except for discontinuities at states representing touch down -- they are "hybrid systems“ [1]. The stability of such systems can be studied by considering the stability of their Poincaré maps [2]. We analyze a class of these hybrid systems in comparison to linear continuous systems with identical Poincaré maps. We compare the robustness to time-varying actuator disturbances of both the hybrid and linear systems through their effect on the Poincaré maps and present the two-dimensional case.

Problem Statement

By restricting trajectories to begin on the line segment S (as shown in Figure 1.) the hybrid system, H

$$H:\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}=\begin{cases}[1-\alpha,1-\alpha]^T: & x>0, \quad y>0\\ [1+\delta_x(t),1-\alpha+\delta_y(t)]^T: & x<0, \quad y>0\\ [1-\alpha+\delta_x(t),1+\delta_y(t)]^T: & x>0 \quad y<0\\ [1,1]^T: & x<0 \quad y<0\end{cases}$$

$$0<\alpha<1$$

and the linear system, L

$$L:\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}=-\frac{\eta}{\sqrt{2}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}+\begin{bmatrix}\frac{1}{\sqrt{2}}+\delta_x(t)\\ \frac{1}{\sqrt{2}}+\delta_y(t)\end{bmatrix}$$

$$\eta=-\log(\sqrt{1-\alpha})$$

define maps from S to the line segment G (also shown in Figure 1.). We then project the output on G back to the input line S and iterate, defining a Poincaré map for the hybrid system:

$$H(\delta_x(t),\delta_y(t)):S\to S$$

and a Poincaré map for the linear system:

$$L(\delta_x(t),\delta_y(t)):S\to S$$

When the disturbances, $\delta_x(t)$ and $\delta_y(t)$, are identically zero, these maps are equivalent. The goal is to analyze the long-term behavior of the maps when disturbances are not identically zero, i.e., determine the following (using a suitable coordinate change):

$$\begin{bmatrix}u\\ w\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

$$\lim_{n\to\infty}H^n(u_0,\alpha,\delta_x(t),\delta_y(t))$$

$$\lim_{n\to\infty}L^n(u_0,\alpha,\delta_x(t),\delta_y(t))$$

$$-1<u_0<1$$

Results

$$\lim_{n\to\infty}H^n(u_0,\alpha,0,0)=\lim_{n\to\infty}L^n(u_0,\alpha,0,0)=0$$

$$\lim_{n\to\infty}H^n(u_0,\alpha,\delta_x(t),\delta_y(t))=0$$

$$\lim_{n\to\infty}L^n(u_0,\alpha,\delta_x(t),\delta_y(t))=\lim_{t\to\infty}\frac{1}{\eta\sqrt{2}}(\delta_x(t)-\delta_y(t))$$

$$-1<u_0<1, 0<\alpha<1$$

The above is true when the actuator disturbances satisfy the following assumptions:

$$0<1-\alpha+\delta_x\le K_1$$

$$0>\frac{1}{\sqrt{2}}(\delta_x-\delta_y-\alpha)\ge -K_2$$

$$0<1-\alpha+\delta_y\le K_3$$

$$0<\frac{1}{\sqrt{2}}(\delta_x-\delta_y+\alpha)\le K_4$$

State Space of the Hybrid and Linear Systems

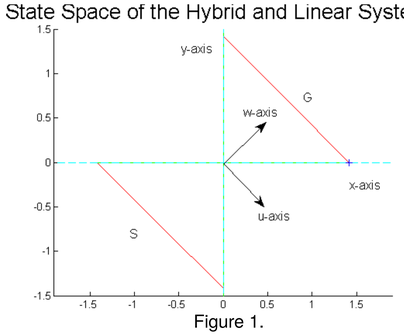

Figure 1.

Hybrid System: (No Disturbances)

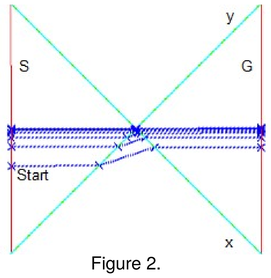

Figure 2.

Linear System: (No Disturbances)

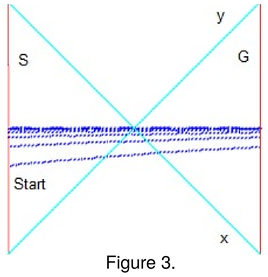

Figure 3.

Conclusions

The results of this research show that, although the undisturbed hybrid and linear systems have identical Poincaré maps, these systems have quite different robustness properties. In particular, our analysis shows that under suitable assumptions, the hybrid system is immune to disturbances whereas the long-term behavior of the iterated linear system depends linearly on the disturbances. We conclude that as a hybrid controller, the hybrid system considered is fundamentally more robust than a comparable linear regulator. This may help explain the widespread appearance of multiple contact gaits in many terrestrial animal species.

$H^n(u_0)$ vs. n (Random Disturbances), u_0 = 0.3

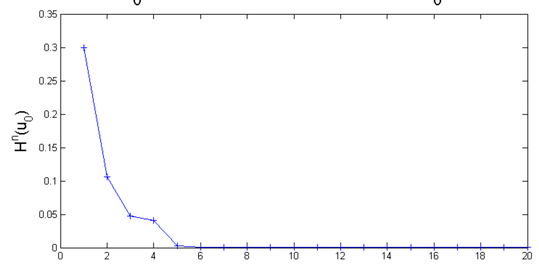

Figure 4.

$L^n(u_0)$ vs. n (Random Disturbances), u_0 = 0.3

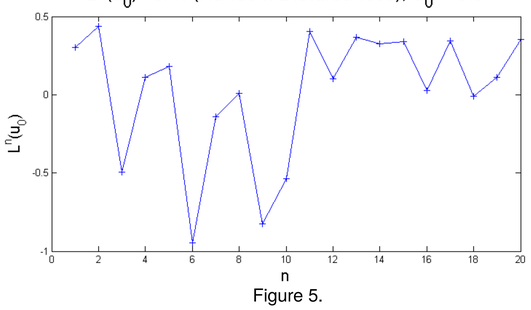

Figure 5.

Acknowledgements

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References

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[2] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcation of Vector Fields*, Springer-Verlag, 1983.

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