Phase-based models of rhythmic systems

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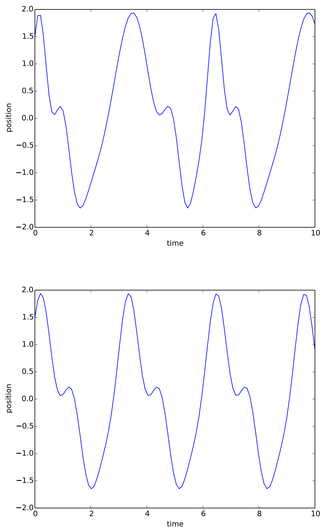
Introduction

- Empirical evidence from legged locomotion data:
  - One stride of an animal gait is very similar to any other stride of the self-same animal
  - Environmental perturbations appear to most strongly affect *phase*.
- Our algorithm normalizes such data sets so that phase changes at a constant rate
  - This allows us to build models with better statistical properties



Assumptions/Background

- Locomotion data can be modeled by rhythmic systems – smooth *oscillators* [4] subject to noise (also applies to many hybrid systems [1])
- Stability basin of an oscillator is partitioned into *isochrons* – codimension-1 manifolds whose points have the same *asymptotic phase*. [3].
- Data-driven algorithms to compute asymptotic phase from time series exist [5].

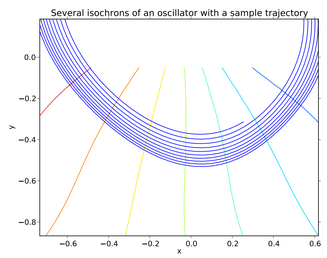
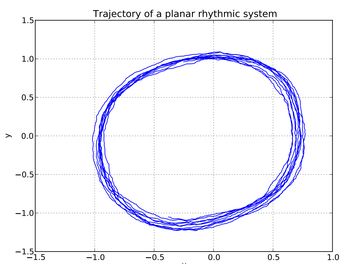
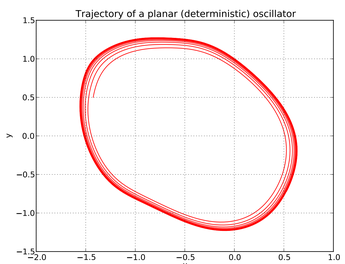


Figure 1: Sample trajectory of a planar oscillator is shown in blue. Points on the same isochron (radial lines) asymptotically meet when flowed forward in time





Limit cycle estimation algorithms

Given: a trajectory $x : [0, M] \to \mathbb{R}^n$, *event* times $E_1, \ldots, E_K \in [0, M]$. **Notation**: $\mathcal{B} \subset \mathbb{R}^n$ is stability basin of oscillator, $\theta : \mathcal{B} \to [0, 2\pi)$ assigns each point its asymptotic phase.

- *Event phase-based estimate* [2]:
  1. $\forall i, \forall \varphi \in [0, 2\pi)$: $\tilde{x}_i(\varphi) := x\left(\varphi \cdot \frac{E_{i+1} - E_i}{2\pi} + E_i\right)$
  2. $\forall \varphi \in [0, 2\pi)$: $e(\varphi) := \sum_{i=1}^{K-1} \tilde{x}_i(\varphi)$
- *Asymptotic phase-based estimate* (new method):
  1. $\forall \varphi \in [0, 2\pi) : D_\varphi := \{t \in [0, M) : \theta(x(t)) = \varphi\}$
  2. $\forall \varphi \in [0, 2\pi)$: $p(\varphi) := \frac{1}{\#(D_\varphi)} \sum_{t \in D_\varphi} x(t)$

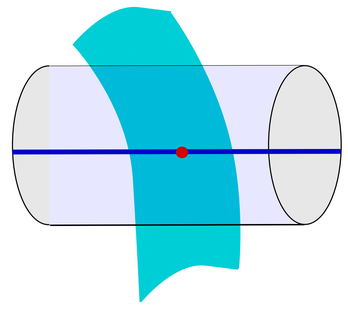


Figure 2: Magnified view of limit cycle shown. **Intuition**: Asymptotic phase-based estimation forces the estimate of $\gamma(\varphi)$ (red dot) to lie on the isochron S_φ (teal surface) – if trajectories lie in a small tube about the limit cycle (blue line), asymptotic phase-based estimation has a small worst-case error.

Experimental Results

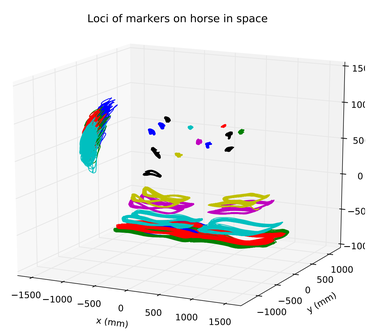


Figure 3: Horse locomotion data courtesy of Simon Wilshin, RVC

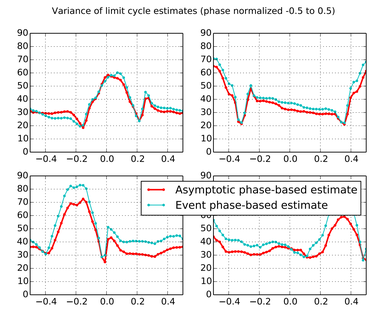


Figure 4: Variance of event phase-based estimate (resp. asymptotic phase-based estimate) is shown in teal (resp. red)

- Plots show variance of both estimates for identically located markers on each of the four legs
- Events used were anterior extreme position (AEP) of marker on front left leg
- Event phase-based estimation to perform reasonably well on front left leg; asymptotic phase-based estimation performs better on all other legs

Theorems

1. For a given oscillator, $\exists g : [0, +\infty] \to [0, +\infty]$ s.t. $\lim_{\epsilon \to 0} g(\epsilon) = 0$ and: if observed trajectories are confined to a tube of radius ϵ of the limit cycle, then $\text{var}(p(\phi)) \leq [g(\epsilon)]^2$ uniformly in φ.
2. Let $\gamma : [0, 2\pi) \to \mathbb{R}^n$ be a parametrization of the limit cycle, let S_φ be the isochron containing $\gamma(\varphi)$. There exist random processes ζ, I_p, I_e s.t. $p(\varphi) = \gamma(\varphi) + I_p(\varphi) \in S_{\gamma(\varphi)}$, $e(\varphi) = \gamma(\varphi + \zeta(\varphi)) + I_e(\varphi) \in S_{\gamma(\varphi + \zeta(\varphi))}$ so that

$$\text{var}[p(\varphi)] = \boxed{\text{var}(I_p(\varphi))}$$

$$\text{var}[e(\varphi))] = \boxed{\text{var}[I_e(\varphi)]} + \text{var}[\gamma(\varphi + \zeta(\varphi))] + \ldots$$

$$\ldots + 2 \cdot \text{trace}(\text{cov}[\gamma(\varphi + \zeta(\varphi)), I_e(\varphi)])$$

Intuition: boxed terms are approximately equal, small. Asymptotic phase-based estimation has small variance – no guarantees for event phase-based estimation!

Conclusions

- Empirically, many data sets appear to experience phase drift.
- We've provided a lower bound on the performance of asymptotic phase-based estimation
- Event phase-based estimation does not satisfy this bound in general.
- Asymptotic phase-based estimation outperformed event phase-based estimation overall on horse data

Acknowledgements

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References

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