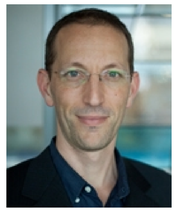
A dynamical systems perspective on Templates & Anchors: some general methods for anchoring templates


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Introduction

- **Template and anchor**: a pair of models
  - Template: "reduced-order" model, describes essential features of biomechanical behavior.
  - Anchor: more detailed, often contains specifics of individual animal morphology [2].
- **Posture principles**: a class of template & anchor pairs
  - If a template is obtained by restriction to subspace of postures, we say a "posture principle" is obeyed [5].
  - Animals have many degrees of freedom, but move "as if" they have only a few.
- **Prior work**: anchoring templates as attracting invariant submanifolds of dynamical systems (e.g., [6]). But do not know (template state) $\leftrightarrow$ (anchor state) correspondence.
- **Our contribution**: general method for constructing such templates while *choosing* this correspondence.

Idea: Model Templates as Normally Hyperbolic Invariant Manifolds (NHIMs)

- (Stable) **NHIM** for flow Φ^t: an exponentially stable invariant manifold M such that the exponential rate of attraction dominates the dynamics tangent to M [1, 5].
- NHIMs have (unique) **asymptotic phase**: any trajectory $\Phi^t(x)$ in the stability basin B is "shadowed" by a unique trajectory $\Phi^t(P(x)) = P(\Phi^t(x))$ in M. (Generalizes isochrons [3])
- **Robustness**: NHIMs persist for small perturbations [1, 5].

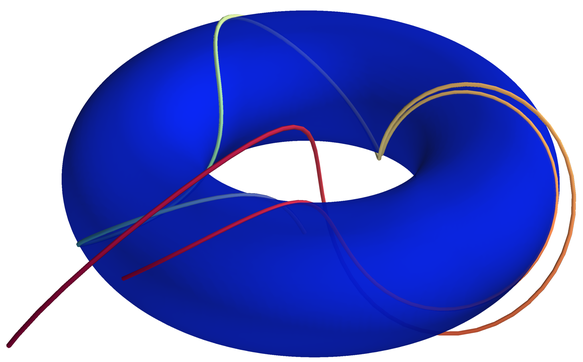


Figure 1: Shown here is a stable, normally hyperbolic torus, constructed using our recipe described in the next section. Torus NHIMs arise naturally as attractors in, e.g., coupled oscillator models. The two trajectories have the same asymptotic phase — after a brief transient, these two trajectories are indistinguishable, i.e., they are "asymptotically in phase."

New Recipe for Template/Anchor Pairs

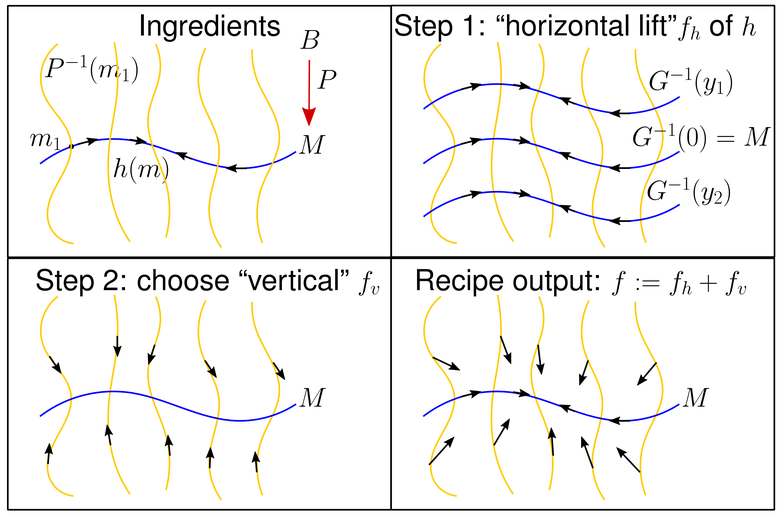


Figure 2: Ingredients: smooth maps $G\colon B \to \mathbb{R}^k$, $P\colon B \to M := G^{-1}(0)$ and template dynamics defined by a vector field $h \in \mathfrak{X}(M)$ on M. Requirements: see handout. P/G level sets shown in yellow/blue.

Result: a stable NHIM M with asymptotic phase P

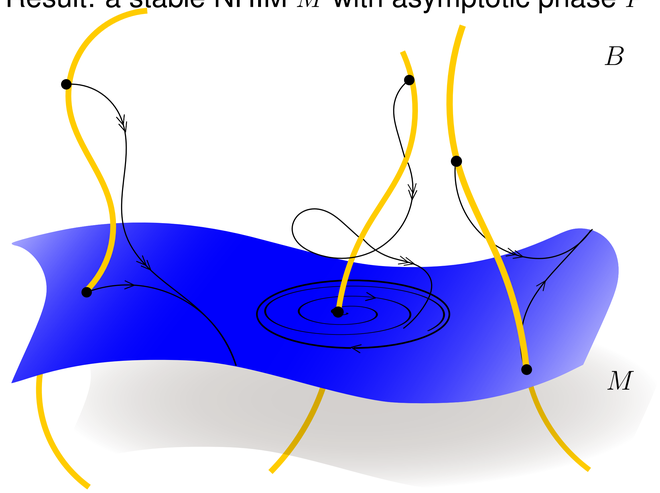


Figure 3: Result of our recipe: an attractive NHIM M having basin of attraction B. Since f stabilizes M, after a brief transient the trajectories emanating from any x and $P(x)$ are indistinguishable.

Our Recipe is Robust and Universal

- **Universality**: can produce *every* NHIM having C^1 phase.
- **Persistence:** Any $\tilde{f}$ near f has nearby $\widetilde{M}, \widetilde{P}$.
- **Structural stability (SS):** (h is SS) $\implies$ (f is SS).

Our Recipe is Robust and Universal

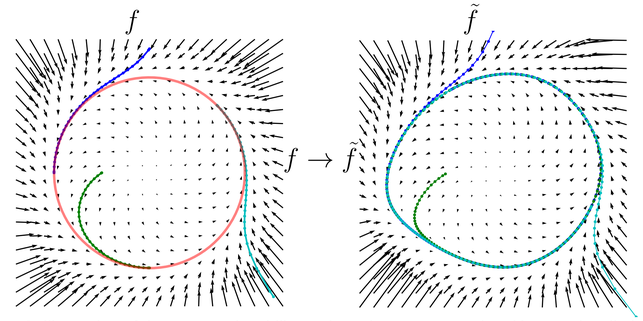


Figure 4: Illustration of the structural stability and persistence properties. Here, M is a limit cycle.

Posture Principle version of the Recipe

- Ingredients: similar, but now $B \subseteq$ (all postures), $h \in \mathfrak{X}(TM)$.
- Output: $f \in \mathfrak{X}(TB)$ with TM a NHIM having phase map DP.
- Can experimentally test if animals employ this recipe.

Conclusion

- Universal method for anchoring structurally stable templates, while choosing the (anchor) $\to$ (template) map.
- Our results suggest an experimentally testable biological hypothesis, and also control system design strategies.
- Future work: strategies for under-actuated systems.

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