Asymptotic phase, model reduction, and control of Templates & Anchors



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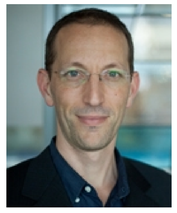

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Introduction

- **Template and anchor**: a pair of models
  - Template: "reduced-order" model, describes essential features of biomechanical behavior.
  - Anchor: more detailed, often contains specifics of individual animal morphology [2].
- **Prior work**: anchoring templates as attracting invariant submanifolds of dynamical systems (e.g., [5]). But do not know (template state) $\leftrightarrow$ (anchor state) correspondence.
- **Our contribution**: methods for constructing such templates while *choosing* this correspondence.

Templates as Normally Hyperbolic Invariant Manifolds (NHIMs)

- **NAIM** (attracting NHIM) for flow Φ^t: an exponentially stable invariant manifold M such that the exponential rate of attraction dominates the dynamics tangent to M [1, 4].
- NAIMs have (unique) **asymptotic phase**: any trajectory $\Phi^t(x)$ in the stability basin B is asymptotic to the trajectory $\Phi^t(P(x)) = P(\Phi^t(x))$ in M. (Generalizes isochrons [3])
- **Robustness**: NAIMs persist for small perturbations [1, 4], stability basin dynamics are structurally stable if NAIM dynamics are.

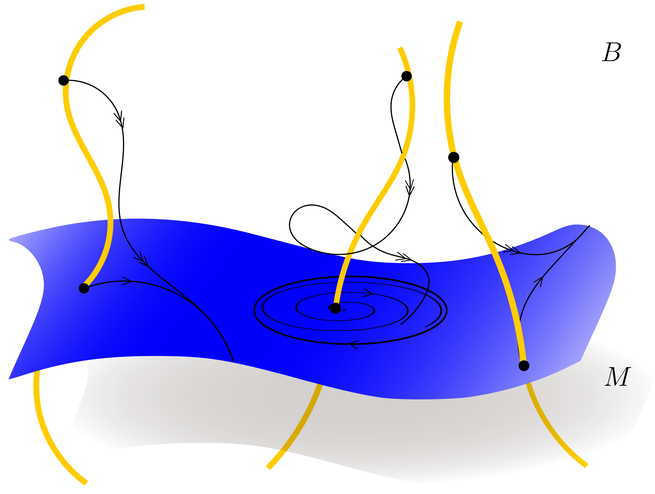

Figure 1: A NAIM M having basin of attraction B. Since f stabilizes M, after a brief transient the trajectories emanating from any x and $P(x)$ are indistinguishable.

NAIMs decomposed

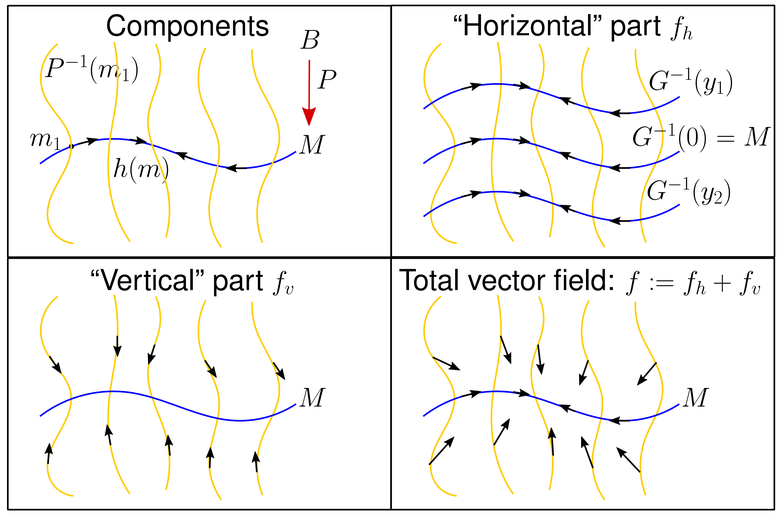

Figure 2: The stability basin B of every NAIM (with trivial normal bundle) is comprised of these components: smooth maps $G\colon B \to \mathbb{R}^k$, $P\colon B \to M := G^{-1}(0)$ and template dynamics defined by a vector field $h \in \mathfrak{X}(M)$ on M. P and G level sets are shown in yellow and blue, respectively.

Control systems affine in control

- Abstract affine control system on B: $\dot{x} = f(x) + u, u \in \mathcal{D}$.
- **Theorem:** Given $M \subset B$ and a submersion & retraction $P\colon B \to M$, there exists a feedback $u(x)$ rendering M asymptotically stable with asymptotic phase P iff
  - $\forall x \in B : DP_x f(x) - f \circ P(x) \in DP_x \mathcal{D}_x$
  - Given $u\colon Q \to \mathcal{D} \subset \mathcal{TQ}$, M is stabilizable for the affine control system $\dot{x} = f(x) + u(x) + \tilde{u}, \tilde{u} \in \mathcal{D} \cap \ker DP$.

Control application: backstepping

- Control system: $\dot{x} = f(x) + B(x)y$, $\dot{y} = u$
- Choose any $\nu(x)$, any smooth Hurwitz family $A(x)$.
- **Theorem:** if $\dim y \geq \dim x$, we can always explicitly find a smooth family of matrices $\phi(x)$ and a feedback control $\tilde{u}(x, y)$ such that the control law

$$u(x,y) = D\nu(x)(f(x) + B(x)\nu(x)) + \frac{1}{\epsilon}A(x)(y - \nu(x)) + \epsilon^2 \tilde{u}(x,y)$$

renders graph(ν) a NAIM with asymptotic phase $P(x, y) = x + \phi(x)(y - \nu(x))$.

Example: on a technique of Koditscheck

- Goal: make a second order control system $\ddot{x} = u$ behave like a first order dynamical system $\dot{r} = \nu(r)$.
- Technique: stabilize the submanifold $\nu(Q) \subset TQ$.
- Using the control law from the last section, we can obtain an exact (velocity-dependent) asymptotic phase map.

Example: oscillator synchronization

- Goal: design/synchronize mechanical nonlinear oscillators
- Design $\dot{r}_i = \nu(r_i)$ to have stable limit cycle with known asymptotic phase, then use last section to lift dynamics to $\ddot{\theta}_i = u(\theta_i)$ — the asymptotic phase maps then compose.
- Now use the uncoupled asymptotic phase of each subsystem for synchronizing controller — can reduce control effort.

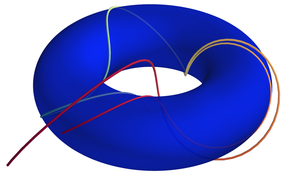

Figure 3: Depiction of the attractor for one of the two (uncoupled) oscillators

Conclusion

- Method for anchoring templates, while choosing the asymptotic phase map.
- Future work: under-actuated mechanical systems.

Acknowledgements

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References

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