Global linearization and fiber bundle structure of invariant manifolds


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Introduction

Normally attracting invariant manifolds (NAIMs) are a generalization of exponentially stable equilibria and periodic orbits. The two properties of NAIMs we need are:

- Existence of stable foliations & local disk bundle structure.
- Local linearizability.

We extend these well-known local results [1, 2, 4, 5, 6] in two ways: we (i) extend the linearization theorem to a class of NAIMs with boundary, the *inflowing NAIMs*, and (ii) we extend all mentioned results to the *global* stable manifold.

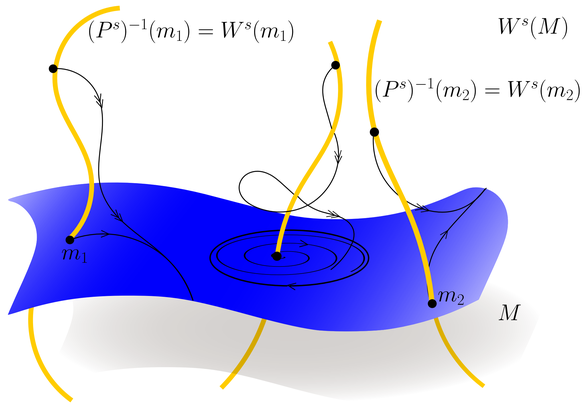


Figure 1: A NAIM M with global stable manifold $W^s(M)$ foliated by global stable fibers $W^s(m)$ and associated projection $P^s\colon W^s(M) \to M$.

Inflowing NAIMs & other rate conditions

Setup: a Riemannian manifold Q, a vector field f with flow Φ^t on Q, a compact submanifold $M \subset Q$ ($\partial M \neq \varnothing$ is allowed). Then M is an inflowing r-NAIM if:

inflowing invariant $\forall t > 0\colon \Phi^t(M) \subset M$ and f points strictly inward at ∂M

$\mathrm{D}\Phi^t$-invariant splitting $\mathsf{T}Q|_M = \mathsf{T}M \oplus E^s$

r-hyperbolic $\|\mathrm{D}\Phi^t|_{E^s_m}\| \leq Ce^{-at}\|\mathrm{D}\Phi^t|_{\mathsf{T}_mM}\|^i$, $0 \leq i \leq r$.

The local foliation $W^s_{\mathsf{loc}}(M) \in C^{k<r}$ if the **center bunching condition** holds [2]:

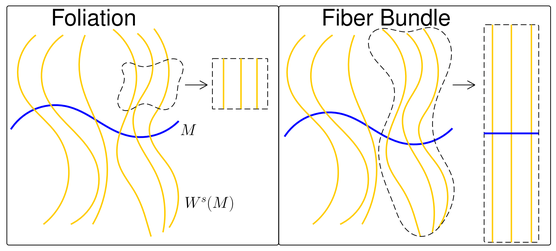
$$\forall t \geq 0\colon \|\mathrm{D}\Phi^t|_{T_mM}\|^i \|\mathrm{D}\Phi^t|_{E^s_m}\| \leq Ke^{-bt}\|\mathrm{D}\Phi^t|_{T_mM}\|, \qquad 0 \leq i \leq k < r.$$

Local linearization [6]: $\Phi^t|_{W^s_{\mathsf{loc}}(M)}$ is C^k conjugate to $\mathrm{D}\Phi^t|_{E^s}$ if for all $t \geq 0$:

$$\begin{aligned} K^{-1}e^{-\delta t} \leq \|\mathrm{D}\Phi^t|_{TM}\| \leq \|\mathrm{D}\Phi^t|_{TM}\| \leq Ke^{\delta t}, \\ K^{-1}e^{-\delta t} \leq \|(\mathrm{D}\Phi^t|_{TM})^{-1}\| \leq \|(\mathrm{D}\Phi^t|_{TM})^{-1}\| \leq Ke^{\delta t}, \\ K^{-1}e^{-\beta t} \leq \|\mathrm{D}\Phi^t|_{E^s}\| \leq \|\mathrm{D}\Phi^t|_{E^s}\| \leq Ke^{-\alpha t}, \end{aligned} \tag{1}$$

with $0 < \delta < \alpha < \beta$ and $K \geq 1$ such that $\alpha > r\delta$, $\beta < 2\alpha - (r-1)\delta$.

Statement of our results



Theorem 1 (Global fiber bundle structure of inflowing NAIMs)
Let $M \subset Q$ be a compact inflowing 1-NAIM for the flow Φ^t generated by the C^r vector field f on Q, and assume that $M \subset Q$ is a C^r submanifold. Further assume that the local projection $P^s_{\mathsf{loc}}\colon W^s_{\mathsf{loc}}(M) \to M$ is C^k, with $0 \leq k \leq r$. Then the global projection $P^s\colon W^s(M) \to M$ defines a C^k fiber bundle structure on the global stable manifold $W^s(M)$. Furthermore, $W^s(M)$ is C^k isomorphic (as a disk bundle) to any C^k vector bundle $\widetilde{E}^s$ over M which C^0-approximates E^s.

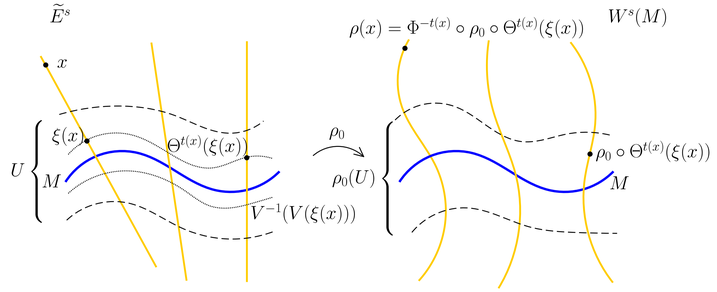


Figure 2: Idea of proof of Theorem 1. Constructing the global fiber-preserving diffeomorphism ρ using the local version ρ_0. Ingredients: V which restricts to a Lyapunov function for $\rho_0^*\Phi^t$ on U, nonlinear parallel transport Θ^t preserving V-level sets, rescaling map ξ which maps V-level sets to V-level sets.

Theorem 2 (Global linearization of inflowing NAIMs)
Let $M \subset Q$ be a compact inflowing r-NAIM for the flow Φ^t generated by the C^r vector field f on Q, where $r \geq 3$. Assume further that the Sakamoto rate conditions (1) *hold. Then $E^s \in C^{r-1}$ and $\Phi^t|_{W^s(M)}$ is globally C^{r-1} conjugate to $\mathrm{D}\Phi^t|_{E^s}$.*

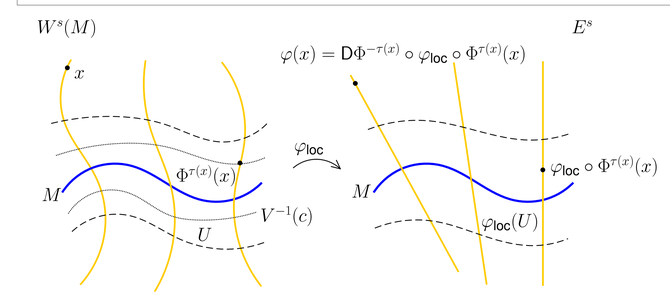


Figure 3: Idea of proof of Theorem 2 for the case $\partial M = \varnothing$. Constructing the global conjugacy φ using the local conjugacy φ_{loc} and a Lyapunov function V for M and the time-to-impact-$V^{-1}(c)$ function $\tau(x)$.

Technique: reduce to the case $\partial M = \varnothing$

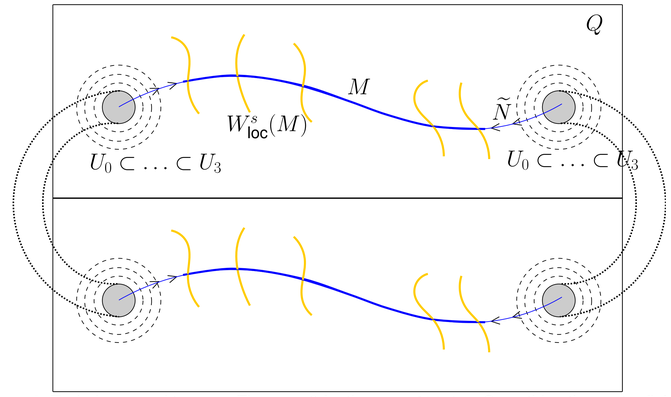


Figure 4: Technique used to prove Theorem 2 for the case $\partial M \neq \varnothing$. Smoothing the vector fields together must be done carefully to preserve exponential rate conditions.

Application to slow-fast systems

Consider the singularly perturbed system

$$x' = f(x, y, \epsilon) \qquad \epsilon y' = g(x, y, \epsilon). \tag{2}$$

For $0 < \epsilon \ll 1$, Theorems 1 and 2 can be applied to find C^{r-1} coordinates $(\tilde{x}, \tilde{y})$ on the **global** stable manifold of an inflowing NAIM for (2) that satisfy either

$$\tilde{x}' = h(\tilde{x}, \epsilon), \qquad \epsilon\tilde{y}' = \Lambda(\tilde{x}, \tilde{y}, \epsilon), \tag{3}$$

or if $\forall (x, y) \in g^{-1}(0)\colon \min \mathrm{Re}(\mathrm{spec}\mathrm{D}_2 g(x, y)) > 2 \max \mathrm{Re}(\mathrm{spec}\mathrm{D}_2 g(x, y))$, then

$$\tilde{x}' = h(\tilde{x}, \epsilon), \qquad \epsilon\tilde{y}' = A(\tilde{x}, \epsilon)\tilde{y}. \tag{4}$$

Acknowledgements

This work was funded by FAPESP grant 2015/25947-6 and ARO grants W911NF-14-1-0573, W911NF-17-1-0243, & W911NF-17-1-0306.

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