Principal Koopman Eigenfunctions for Nonlinear and Nonsmooth Systems



Matthew Kvalheim [1]



Shai Revzen [2]

[1]University of Michigan, Dept. of EECS [2]University of Michigan, Depts. of EECS, EEB, and Robotics

Introduction

- Koopman theory is closely related to *linearizing coordinates* for dynamics.
- For invertible linear $A\colon \mathbb{R}^n \to \mathbb{R}^n$, Mohr and Mezić [2] defined the *principal Koopman eigenfunctions* to be those of the form $x \mapsto \langle w, x\rangle$ with w a left eigenvector of A; similarly for $(x,t) \mapsto e^{tA}x$.
- We generalize this definition to classes of non(linear/smooth) systems.
  - For a flow Φ on a manifold Q having an attracting invariant subset M, say an eigenfunction $\psi \in C^1$ is *principal* if $\psi|_M \equiv 0$ and $\forall x \in M\colon d\psi(x) \neq 0$.
  - For $Q = \mathbb{R}^n$ and ψ a Bouligand (B-)differentiable eigenfunction, replace $d\psi(x)$ with $\psi'(x,\cdot)$ in latter condition.
  - We prove uniqueness results for principal Koopman eigen(values/functions).
- Motivation for nonsmooth systems: legged locomotion.
  - Many locomotion models are nonsmooth but locally conjugate to smooth systems [1]
  - This reduces eigenfunction uniqueness problem to that of nonsmooth eigenfunctions for smooth systems (c.f. Thm 2).

Bouligand (B-)differentiablity [3]

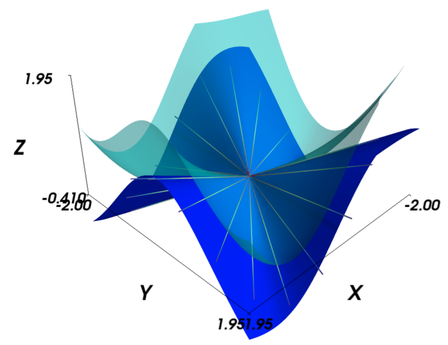

Figure 1: Teal surface is graph of B-differentiable function; blue is graph of its B-derivative..

Sternberg condition and spectral spread

Definition 1 (Sternberg condition of order N for flows [4]) *Let $A \in \mathrm{GL}(n,\mathbb{C})$ with eigenvalues $\lambda_1,\dots,\lambda_1$ repeated with multiplicities, and $\sigma(A)$ the spectrum of A. Let $m = (m_1,\dots,m_n) \in \mathbb{N}^n$, and define*

$$\gamma(\lambda, m) \coloneqq \lambda - (m_1\lambda_1 + \cdots + m_n\lambda_n)$$

where $\lambda \in \mathbb{C}$. Let $|m| \coloneqq m_1 + \dots m_n$. A satisfies the Sternberg condition of order N*, $2 \leq N \leq +\infty$, if $\gamma(\lambda, m) \neq 0$ for all $\lambda \in \sigma(A), m \in \mathbb{N}^n$ s.t. $2 \leq |m| \leq N$.*

Definition 2 (Spectral spread for flows [4]) *Let $A \in \mathrm{GL}(n,\mathbb{C})$ having only eigenvalues $\{\lambda_1,\dots,\lambda_n\}$ with negative real part. Define the* spectral spread *$\nu(A)$ of A to be $\nu(A) \coloneqq \max_{i,j\in\{1,\dots,N\}} \frac{|\mathrm{Re}(\lambda_i)|}{|\mathrm{Re}(\lambda_j)|}$.*

Statement of our results

Theorem 1 (Uniqueness of $C^{k,\alpha}$ principal eigenfunctions) *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ a C^1 vector field having 0 as a globally exponentially stable (GES) equilibrium. Fix $k \in \mathbb{N} \cup \{+\infty\}$ and $0 \leq \alpha \leq 1$, assume that $\mathrm{D}_0 f$ has distinct eigenvalues, satisfies the Sternberg condition of order k, and that $\nu(\mathrm{D}_0 f) < k + \alpha$. Let φ_1, φ_2 be two C^k principal eigenfunctions with eigenvalue λ. Then $\lambda \in \mathrm{spec}(\mathrm{D}_0 f)$ and there exists $c \in \mathbb{C}$ s.t. $\varphi_1 = c\varphi_2$.*

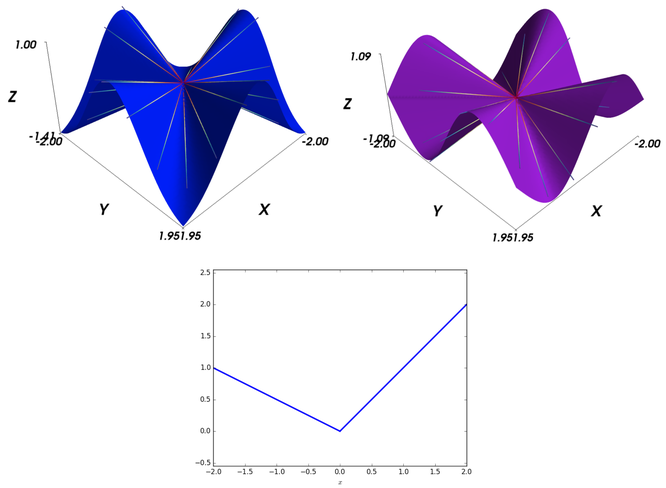

Figure 2: Theorem 2 implies that the B-derivative of a principal eigenfunction with complex eigenvalue is given by projection onto an eigenspace followed by a map whose real and imaginary parts are of the form shown in the top two figures ($k = 4$ shown). For the real case, the B-derivative is given by a projection followed by a positive homogeneous map as shown in the bottom figure..

Theorem 2 (Loc. Lipschitz & B-differentiable principal eigenfunctions) *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ a C^1 vector field having 0 as GES equilibrium. Assume $\mathrm{D}_0 f$ has distinct eigenvalues, and furthermore assume that any pair of eigenvalues of $\mathrm{D}_0 f$ have distinct real parts. Let ψ a loc. Lipschitz B-differentiable principal eigenfunction with eigenvalue λ. Then $\exists \mu \in \mathrm{spec}(\mathrm{D}_0 f)$, $k \in \mathbb{Z}$ s.t.*

$$\lambda = \mathrm{Re}(\mu) + ik\mathrm{Im}(\mu). \tag{1}$$

Let $V \in \mathrm{GL}(n,\mathbb{R})$ be s.t. $V^{-1}(\mathrm{D}_0 f)V$ is a real canonical form. Then furthermore, if $\mathrm{Im}(\mu) = 0$ there exist constants $c_{-1}, c_1 \in \mathbb{C}$ s.t.

$$\forall x \in \mathbb{R}^n\colon \psi'(0;x) = c_{\mathrm{sgn}\langle V^{-1}e_\mu, V^{-1}x\rangle} |\langle V^{-1}e_\mu, V^{-1}x\rangle|, \tag{2}$$

where $e_\mu \in \mathbb{R}^n$ is the real eigenvector of $\mathrm{D}_0 f$ with eigenvalue $\mu \in \mathbb{R}$. If instead $\mathrm{Im}(\mu) \neq 0$, then $\exists c \in \mathbb{C}$, $k \in \mathbb{Z}$ s.t.

$$\forall x \in \mathbb{R}^n\colon \psi'(0;x) = c\,|z_{\mu,x}|\, e^{ik\arg(z_{\mu,x})}, \tag{3}$$

where

$$z_{\mu,x} \coloneqq \langle V^{-1}\mathrm{Im}(e_\mu), V^{-1}x\rangle + i\langle V^{-1}\mathrm{Re}(e_\mu), V^{-1}x\rangle \in \mathbb{C}, \tag{4}$$

and where $\mathrm{Im}(e_\mu), \mathrm{Re}(e_\mu) \in \mathbb{R}^n$ are columns of V corresponding to a complex eigenvector $e_\mu \in \mathbb{C}^n$ of $\mathrm{D}_0 f$ corresponding to the eigenvalue $\mu \in \mathbb{C}$.

Example: Theorem 1 is sharp

$$\dot{x} = -x \qquad \dot{y} = -(\pi - 3 + k)y. \tag{5}$$

Both $h_1(x,y) \coloneqq y$ and $h_2(x,y) \coloneqq y + |x|^{\pi-3+k}$ are $C^{k,\pi-3}$ principal Koopman eigenfunctions for (5) having the same Koopman eigenvalue $\lambda = -(\pi - 3 + k)$.

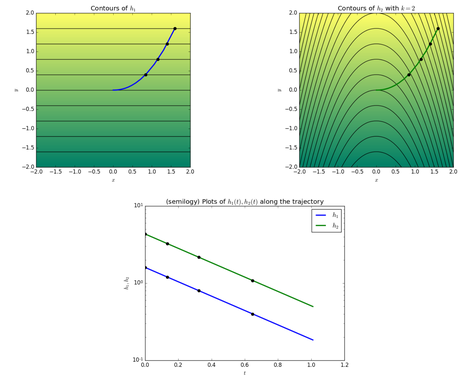

Figure 3

Discussion

- Easy to see analytic principal eigenfunctions unique; $C^{k,\alpha}$ case more subtle.
- Theorem 1 is relevant for rigorous use of piecewise-polynomial interpolants to numerically compute eigenfunctions.
- Theorem 2 is a first step towards a Koopman theory relevant for dissipative nonsmooth systems in robotics, biology, etc.
- Not shown here: we have generalized our results to attracting periodic orbits and periodic points for discrete dynamical systems.
- Future work: more general attracting invariant manifolds.

Acknowledgements

This work was funded by an ARO MURI W911NF-17-1-0306 (sub-award from U. Cal. Santa-Barabara) “From Data-Driven Operator Theoretic Schemes to Prediction, Inference, and Control of Systems.”

References

[1] S A Burden, S S Sastry, D E Koditschek, and S Revzen. Event–selected vector field discontinuities yield piecewise–differentiable flows. *SIAM Journal on Applied Dynamical Systems*, 15(2):1227–1267, 2016.

[2] R Mohr and I Mezić. Koopman principle eigenfunctions and linearization of diffeomorphisms. *arXiv preprint arXiv:1611.01209*, 2016.

[3] S Scholtes. *Introduction to Piecewise Differentiable Equations*. Springer Science & Business Media, 2012.

[4] G R Sell. Smooth linearization near a fixed point. *American Journal of Mathematics*, pages 1035–1091, 1985.

[5] S Sternberg. Local contractions and a theorem of Poincaré. *American Journal of Mathematics*, 79(4):809–824, 1957.