Saltation Matrices for Multi-Contact Robot Dynamics

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Robots Live by Contact

Contact is how a robot acts on the world. Legged machines, manipulators, sky-cranes and cars all depend on it -- and are built to keep it simple: quadrupeds drive a diagonal leg pair as one leg, tires stay non-slip, no commercial legged robot has toes.

The obstacle is mathematical. Planning and control rest on perturbation analysis which typically uses derivatives. Contact introduces jump discontinuities that invalidate derivatives; that trouble compounds when several contacts form in an unknown order.

Terminology

hybrid dynamical system - Here an ODE whose right-hand side is smooth except on finitely many surfaces in the state space.

guard - one such surface. Here: one contact made or broken.

mode - the smooth vector field between crossings, i.e. one fixed set of active contacts.

mode schedule - the sequence order in which modes are used. N contacts admit N! orders.

flow - the map from an initial state to the state t seconds later.

flow Jacobian - derivative of the flow in its initial condition. Linearized control, Kalman filtering and orbital stability all use it.

saltation matrix - the correction to the flow Jacobian at a guard crossing.

transverse crossing - crossing at a speed bounded away from zero: no grazing, no rebound.

Poincare section - a surface the flow crosses transversely multiple times. Here the the touchdown map is defined a map between input and output Poincare sections.

The Combinatorial Problem

- Hard-contact impacts annihilate the penetrating component of velocity, leaving a discontinuous right-handside.
- Under simultaneous contact the flow need not be continuous, let alone differentiable (Remy 2010). Each mode schedule is then analysed separately and cost grows combinatorially.
- Yet almost every terrestrial animal uses many gaits with near simultaneos contacts..

KEY RESULT

When Forces depend on Position alone and are an Additive System (FoPAS): the flow through a multi-contact event is C ^{1}, and all N! mode schedules share one first-order model. **That combinatorial explosion never reaches the linearization.**

Background: Saltation

Between contacts the flow Jacobian obeys the variational equation. At a transverse crossing of one h(x)=const. guard, where the vector field changes from F(x$_-$) to F(x$_+$) it is corrected by the *saltation matrix*

$$S(x) = I + \frac{(F(x_+) - F(x_-)) \otimes \nabla h(x)}{\nabla h(x) \cdot F(x_-)}$$

With several guards the correction is the product of the individual matrices, product taken in crossing order. Different orders, different products: the flow Jacobian is only piecewise linear.

Event Selected System (ESS)

- [x] Trajectories are continuous
- [x] Contacts appear at smooth surfaces in the state space
- [x] All contacts are (locally) made or broken exactly once
- [x] Trajectories flow transversely through the guards (no scuffing or bouncing back)

Force-Position Additive System (FoPAS)

- [x] Is an ESS
- [x] States split into *position* and *velocity* coordinates
- [x] Contacts are functions of the position variables only
- [x] Contacts introduce additive accelerations

Theorem: FoPAS Flows are C ^{1}

At a multi-contact point of a FoPAS the saltation update is the **sum**, not the product, of the individual terms:

$$S(\rho) = I + \sum_{k=1}^{N} \frac{\delta F_k \otimes \nabla h_k}{\nabla h_k \cdot F_k^{(-)}}$$

Proof. Split the state as (x, v). A guard depends on position alone; a contact adds an acceleration. Their blocks are complementary:

$$\nabla h_k = (n_k, 0), \quad \delta F_k = (0, \delta F_{k,v})$$

So each single-contact saltation matrix is block unitriangular, with the normal approach speed in the denominator:

$$S_k = \begin{bmatrix} I & 0 \\ A_k & I \end{bmatrix}, \; A_k = \delta F_{k,v} \otimes n_k \,/\, (n_k \cdot v)$$

Any two off-diagonal blocks annihilate: $(n_j, 0) \cdot (0, \delta F_{k,v}) = 0$. Every higher-order term of the product carries such a factor, so for **every** permutation σ:

$$S_{\sigma(N)} \cdots S_{\sigma(1)} = \begin{bmatrix} I & 0 \\ \Sigma_k A_k & I \end{bmatrix}$$

The sum commutes: all N! mode schedules give the same matrix. The flow Jacobian is linear and the flow is C ^{1}, whatever the number of contacts. ∎

Experimental Validation: 3-Legged Hopper

Three springy legs about the body z-axis, preloaded by a cable so they push the instant they touch. **Hypothesis: the linearizations of all 6 mode schedules are identical**

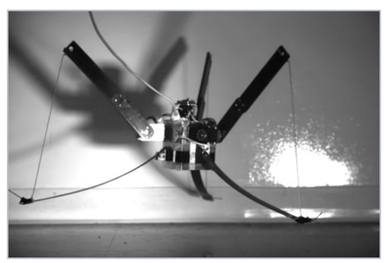

Fig. 1. Drop tests rebound above 40% of release height: the collision is not plastic and the dynamical state survives touchdown, unlike deadbeat hoppers whose flow Jacobians are zero.

Data and Method

- 100 Hz over 4 days: 182,611 frames, 2,860 usable hops, 0.6 mm marker noise.
- Poincare sections are body-Z level sets bracketing touchdown: all feet up on the input section, all down on the output section.
- Mode schedule from the order of foot z-velocity minima; floor height varies by a few mm, so foot height is less reliable.

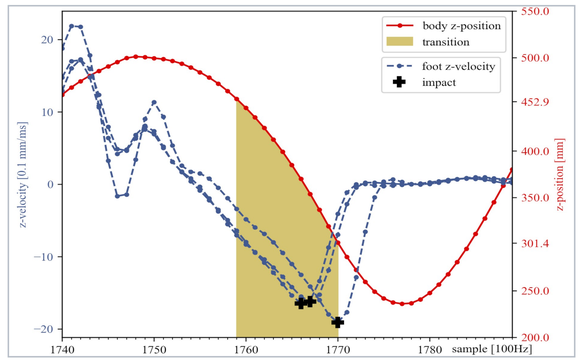

Fig. 2. Foot z-velocity minima (crosses) within the transition (shaded) give the mode schedule.

Results

Models specialized to each mode schedule predict no better than one unspecialized model -- the signature of a C^1 flow. Both beat their mean predictors by a wide margin.

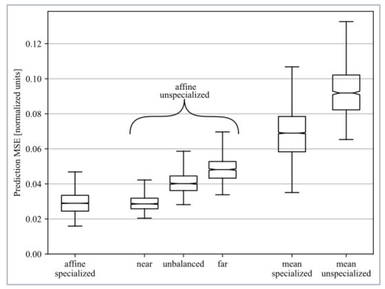

Fig. 3. Bootstrapped prediction MSE. Specializing by mode schedule buys nothing; restricting to touchdowns near the triple contact does help, so higher-order terms differ.

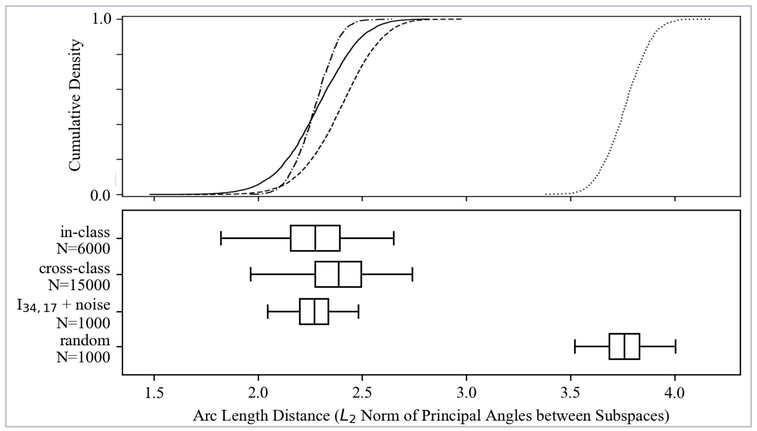

Fig. 4. Geodesic distances between bootstrapped subspaces. In-class and cross-class overlap, and match identical matrices at 10:1 SNR; random subspaces sit far away.

Conclusions

One affine map, ignorant of mode schedule, describes the triple collision. About 100 touchdowns should suffice; we used 3,000 to establish the result statistically.

Linear control and gradient-based optimization should work on multi-contact trajectories nearly out of the box.